Revista de Ciencias Tecnológicas (RECIT). Universidad Autónoma de Baja California ISSN 2594-1925
Volumen 5 (1): e214. Enero-Marzo, 2022. https://doi.org/10.37636/recit.v5n1e214.

Research article

Master-slave synchronization in the Rayleigh and Duffing oscillators via elastic and dissipative couplings

Sincronización maestro-esclavo en los osciladores Rayleigh y Duffing mediante acoplamientos elásticos y disipativos

Ulises Uriostegui-Legorreta, Eduardo Salvador Tututi-Hernández

Facultad de Ciencias Físico-Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Av. Francisco J. Mújica S/N, C.P. 58060, Morelia, Michoacán, México

Corresponding author: Ulises Uriostegui Legorreta, Facultad de Ciencias Físico-Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Av. Francisco J. Mújica S/N, C.P. 58060, Morelia, Michoacán, México. E-mail: uli_mat@hotmail.com. ORCID: 0000-0001-9905-6060

Received: November 17, 2021 **Accepted:** February 4, 2022 **Published:** February 16, 2022

Abstract. - *In this work a master-slave configuration to obtain synchronization between the Rayleigh and the Duffing oscillators is studied. For this configuration, we analyze the system when the dissipative coupling and one that combines the elastic and dissipative couplings are used. We analyzed the coupling parameters to find the range where synchronization between the oscillators is achieved. We found synchronization in the oscillators for large values of the coupling parameter. Our numerical findings show that for the dissipative coupling, there exists partial synchronization while for the others there is complete synchronization.*

Keywords: Nonlinear dynamics; Control of chaos; Synchronization.

Resumen. - En este trabajo se estudia una configuración maestro-esclavo para obtener sincronización entre los osciladores Rayleigh y Duffing. Para esta configuración, analizamos el sistema cuando se utiliza el acoplamiento disipativo y uno que combina los acoplamientos elástico y disipativo. Analizamos los parámetros de acoplamiento para encontrar el rango donde se logra la sincronización entre los osciladores. Encontramos sincronización en los osciladores para valores grandes del parámetro de acoplamiento. Nuestros hallazgos numéricos muestran que para el acoplamiento disipativo existe una sincronización parcial mientras que para los demás existe una sincronización completa.

Palabras clave: Dinámica no lineal; Control del caos; Sincronización.

1. Introduction

Since the seminal work of Pecora and Carroll on synchronization [1], numerous works on chaos that comprises diverse areas such as lasers, chemical reactions, electronic circuits, biological systems, among others, have been studied. In particular, low-dimensional systems have been of interest in order to understand the synchronization and chaotic behavior in nature. The most studied and representative systems are the Lorenz, Chua, Rössler, van der Pol, Rayleigh, Duffing and their variations [2-7].

The Rayleigh oscillator is much like the van der Pol oscillator. The Rayleigh and Duffing oscillators are the paradigmatic circuits to study chaos in systems of low-dimensional. The first gives a limit cycle and the last provides the prototype of a strange attractor. It is well known, that a limit cycle is a closed trajectory in phase space having the property that at least one other trajectory spirals into it, when $t \to \pm\infty$. In other words, the limit cycle is an isolated trajectory; it spirals either towards or away from the limit cycle. An attractor is called strange if it has a fractal structure. This is often the case when the dynamics on the attractor is chaotic. If a strange attractor is chaotic, it exhibits sensitive dependence on the initial conditions. Studies focused on the Rayleigh oscillator reveal that it possesses a rich dynamical structure, especially when the oscillator is forced. This system exhibits complex bifurcation structures with an important number of periodic states, a chaotic region and islands of periodic states, showing, in addition, transitions from chaos to stable states. The dynamics based on identical or distinct linear oscillators presenting the same kind of attractors is still under study [8,9]. Nevertheless, the dynamics of these systems in states of different attractors is of current interest and it could give rise to important information. The control of chaos is concerned with using some designed control to modify the characteristics of a nonlinear system. A number of methods such as active control, adaptive control, optimal control and sliding mode control exist for the control of chaos in systems [10-13]. Several kinds of synchronization play an important role in the study of chaos such as phase synchronization, anticipated synchronization, generalized synchronization, projective synchronization, that have been studied and applied to a chaotic and unified system by *J. Yan et. al* [14]. *A. Razminia et. al* [15] have obtained complete synchronization in chaotic systems of fractional order through sliding mode control. *A. Ouannas et. al* [16] present new approaches to study coexistence of some kinds of synchronization between hyperchaotic systems such as hybrid synchronization and anti-synchronization. *E. Campos et. al* [17] analyzed the multimodal synchronization on the master-slave configuration. *J.S. González et. al* [18] studied the synchronization between two different coupled chaotic oscillators with an external force. The itinerary synchronization between piecewise linear systems with different number of attractors was studied by *A. Anzo-Hernández et. al* [19]. The hybrid function projective synchronization of chaotic systems has been developed and used on systems where the parameters of the system are unknown by applying adaptive control, *A. Khan et. al* [20]. *A. Karimov et. al* [21] have studied the adaptive generalized synchronization between an analog circuit and a computer model by comparing the numerical methods used on the computer simulation of chaotic systems.

Some applications of the Rayleigh and Duffing oscillators go from physics to biology, electronics, chemistry and many other fields. For instance, a possible application of synchronization in chaotic signals is to

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implement secure communication systems, since chaotic signals are usually broadband, noise like, and difficult to predict the behavior. They can also be used for masking information bearing waveforms [22,23,24]. In robotics, the oscillators have been included to control joint hips and knees of human-like robots to ensure the mechanical system follows the right path. The generated signals can be used as reference trajectories for the feedback control [25,26]. Other application is in artificial intelligence. In fact, the oscillators have shown usefulness to training neuronal network and recognition of chaotic systems [27,28].

As far as the coupling between the Rayleigh and Duffing oscillators is referred, we can mention three different couplings, namely: gyroscopic, dissipative and elastic [29-34]. Among the diverse way of coupling, the most used are the elastic and dissipative ones [34,35,36]. In a previous work [34], it is analyzed a different approach of synchronizing two distinct oscillators of low-dimensional, using the aforementioned couplings. *Uriostegui et. al* [37] studied synchronization between the van der Pol and Duffing oscillators by using the elastic, dissipative and a combination of both couplings. It was found that the elastic coupling leads to no synchronization, whilst with the dissipative one it is reached partial synchronization. For the combination of both couplings, it is reached complete synchronization.

In this work, we study and compare two types of couplings by using the Rayleigh and Duffing systems: the dissipative and the used previously by *Uriostegui et. al* [34]. It is important to remark that the studies in the literature on this kind of synchronization is based only on one coupling. An outline of this work is as follows. In Sec. 2, it is briefly studied the main features of the Rayleigh and Duffing oscillators. In Sec. 3, we study and compare two types of couplings using the Rayleigh and Duffing systems upon the master-slave configuration. In Sec. 4, some conclusions and an outlook are presented.

2. Dynamics of the oscillators

The dynamics of the forced Rayleigh oscillator is described by the following nonlinear differential equation:

$$\ddot{x} - \mu(1 - \dot{x}^2)\dot{x} + \frac{dU_1}{dx} = A_1 \cos(\omega_1 t), \qquad (1)$$

The Rayleigh oscillator is characterized by nonlinear damping. The x variable denotes the position, t the time, and $\mu > 0$ is a parameter that governs the nonlinearity and damping. The external forcing is given by the harmonic function, with amplitude A_1 and frequency ω_1. We have defined the function:

$$U_1(x) = \frac{1}{2}x^2. \qquad (2)$$

as the Rayleigh potential, which represents a single-well (see Fig. 1 (a)). The potential has a minimum located at $x = 0$. In order to express Eq. (1) as a dynamical system and to analyze the fixed points, we set $\dot{x} = u$ and drop the forcing to obtain

$$\begin{aligned} \dot{x} &= u, \\ \dot{u} &= \mu(1 - u^2)u - x. \end{aligned} \qquad (3)$$

We can observe from Eq. (3) that the only fixed point is located at $(x = 0, u = 0)$. For the case when $A_1 = 0$, the Rayleigh oscillator satisfies the Rayleigh-Liénard theorem, giving a limit cycle in the phase space, around the origin.

On the other hand, the Duffing oscillator is a nonlinear dynamical system governed by

$$\ddot{y} + \alpha\dot{y} + \frac{dU_2}{dy} = A_2 \cos(\omega_2 t), \qquad (4)$$

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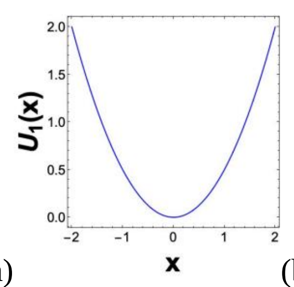
where

$$U_2(y) = -\frac{1}{2}y^2 + \frac{1}{4}\varepsilon y^4. \qquad (5)$$

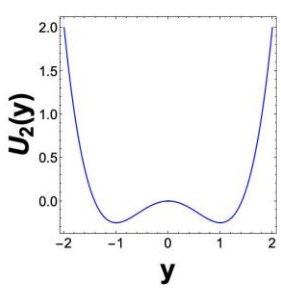
and α is positive and it denotes a dissipative parameter, ε is a positive constant that controls the nonlinearity of the system, and A_2 is the amplitude of the external forcing, being ω_2 its frequency. The potential in Eq. (5) represents a double-well shown in Fig. 1 (b). The local minima of this potential are located in $y = \pm\frac{1}{\sqrt{\varepsilon}}$ and the local maximum is located at $y = 0$. As a dynamical system the Duffing equation in (4) (no forcing) can be cast as

$$\begin{aligned} \dot{y} &= v, \\ \dot{v} &= -\alpha v + y - \varepsilon y^3, \end{aligned} \qquad (6)$$

where we set $\dot{y} = v$. The fixed points for this system are located in the phase space at $(y = 0, v = 0)$ and $\left(y = \pm\frac{1}{\sqrt{\varepsilon}}, v = 0\right)$. The first one at $(y = 0, v = 0)$ is a saddle point, while the others, depending on the parameter α, they can be stable or unstable points. For $\alpha > 0$ the points result stables, for the $\alpha = 0$ case, the resulting dynamics is of type center and for $\alpha < 0$ case, the points result unstable. In particular, when the damping is positive $(\alpha > 0)$, the trajectory of the system is spiral stable, conversely, for a damping negative $(\alpha < 0)$, the trajectory is spiral unstable at the fixed points $\left(y = \pm\frac{1}{\sqrt{\varepsilon}}, v = 0\right)$ in both cases.



(a)



(b)

Figure 1. The potentials $U_1(x)$ and $U_2(y)$. (a) The potential corresponds to the Rayleigh oscillator. (b) The Duffing oscillator (ε = 1).

3. Master-slave synchronization

In this section, two different couplings for the Rayleigh and Duffing systems are studied and compared among themselves, namely: the dissipative and the one that combines elastic and dissipative couplings employed by *Uriostegui et. al* [34]. Let us stress that most of the research on synchronization is based on autonomous systems of three-dimensional or higher [38,39,40]. Three of the most studied nonautonomous systems of low-dimensional with forcing are the Duffing, van der Pol, Rayleigh and their variations, since much of the dynamical features embedded in the physical systems can be realized on these systems [41,42,43]. One important implication is that a two-dimensional continuous dynamical system cannot give rise to strange attractors. In particular, chaotic behavior arises only in continuous three-dimensional dynamical systems or higher. Most of the research on synchronization is based on autonomous systems that satisfy the Poincaré-Bendixson theorem. Nevertheless, let us stress that the Rayleigh and Duffing oscillators being two-dimensional, need an external forcing to present chaos.

The dynamics for each oscillator under study is described by the equations in (1) and (4). The values of the parameters we use are as follows: μ=1.2, α=0.3, ε=1, A_1 =2.8, ω_1 =0.2, A_2 =0.5 and ω_2 =1.3. In Figs. 2 and 3 it is displayed the respective trajectories with the initial conditions x (0) = 1, y (0) = 2, u (0) = 1 and v (0) = -1. Let us mention that the very same values of the parameters and the initial conditions will be used in the subsequent numerical simulations. The numerical simulations were performed using the fourth order of the Runge–Kutta method.

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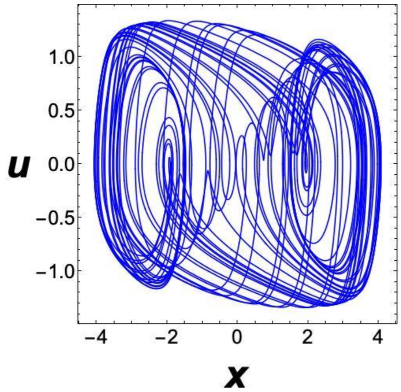

Figure 2. Rayleigh oscillator described by Eq. (1).

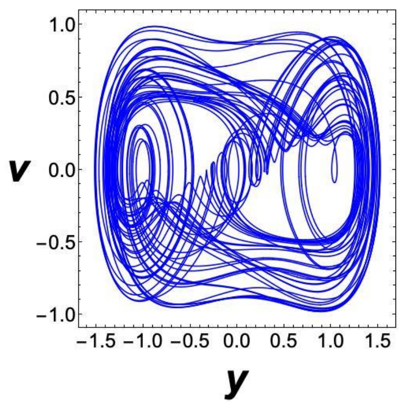

Figure 3. Duffing oscillator described by Eq. (4).

In the configuration master-slave, the Duffing oscillator acts as master and the Rayleigh oscillator as slave. For this case we have

$$Master\colon \begin{cases} \dot{y} = v, \\ \dot{v} = -\alpha v + y - \varepsilon y^3 + A_2\cos(\omega_2 t), \end{cases} \quad (7)$$

$$Slave\colon \begin{cases} \dot{x} = u, \\ \dot{u} = \mu(1-u^2)u - x + A_1\cos(\omega_1 t) + H(v-u). \end{cases} \quad (8)$$

In this instance, the coupling used corresponds to a dissipative coupling and it is represented by $H(v-u)$, being H a coupling parameter to be varied. For the $H=0$ case, the system decouples. The coupling is linear feedback to the slave oscillator and it can be seen as a perturbation for each oscillator in the system, proportional to the difference of the velocity, named in literature a dissipative coupling. We are interested in studying how the dynamics of the system evolves as the constant coupling H changes.

In general, the synchronization problem reduces to finding a suitable value of the coupling strength H, (denoted by H^*) being in the range $H \geq H^* > 0$, such that the master and slave systems synchronize. Thus, for a coupling strength H^*, when the complete synchronization is reached, the error function goes to zero:

$$\lim_{t\to\infty}|y(t)-x(t)| = \lim_{t\to\infty}|v(t)-u(t)| = 0. \quad (9)$$

When the system is in practical synchronization, for a certain value of H^*, the error functions satisfy

$$\lim_{t\to\infty}|y(t)-x(t)| \leq \delta, \quad (10)$$

$$\lim_{t\to\infty}|y(t)-x(t)| \leq \tau, \quad (11)$$

for given positive values $\delta, \tau > 0$, and arbitrary initial conditions. This definition is used, because, sometimes, the errors do not exactly converge to zero, but in practice we can still speak of synchronized systems. In some cases, it can be reached complete synchronization in a single state of the system while in the other, it can be only obtained practical or null synchronization. Although, in practice, such as in analog circuits, we have no total control on the parameters used (e.g., resistors, capacitors, transistors), which makes not possible reproducing the required conditions in the numerical simulations. This could give no complete synchronization. The partial synchronization is the phenomenon when, in a dynamical system, only part of the state variables synchronizes and the others do not.

Let us consider the error functions $|y(t)-x(t)|$ and $|v(t)-u(t)|$ by taking H as a control parameter to be varied in small steps from 0 to 200. For our case, the error functions allow us to

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find the range of values for H in which the synchronization is reached in the projections onto the (x, y) and (u, v) planes, as it can be shown in Figs. 4 and 5. Notice that in the projection onto the (x, y) plane no complete synchronization exists since the error function $|y(t) - x(t)|$ do not exactly converge to zero; the $|v(t) - u(t)|$ function goes to zero for large values of H. For the projection onto the (u, v) plane, the complete synchronization could be reached for rather large values of H. In order to see this, let us observe that the errors $e_1 = y - x$ and $e_2 = v - u$ can be calculated from Eqs. (7) and (8) as:

$$\dot{e}_1 = \dot{y} - \dot{x} = e_2,$$
$$\dot{e}_2 = \dot{v} - \dot{u} = -\alpha v + y - \varepsilon y^3 + A_2 \cos(\omega_2 t)$$
$$-\mu(1 - u^2)u + x - A_1 \cos(\omega_1 t) - H(e_2). \qquad (12)$$

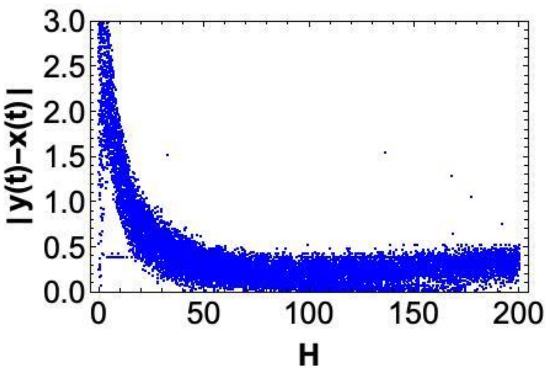

Figure 4. The error function $|y(t) - x(t)|$ varying the parameter H.

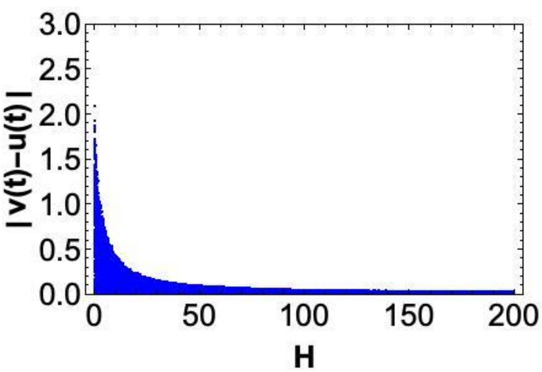

Figure 5. The error function $|v(t) - u(t)|$ varying the parameter H.

The plots of $|e_1|$ and $|e_2|$ as a function of t for a value of $H = 200$, are depicted in Fig. 6.

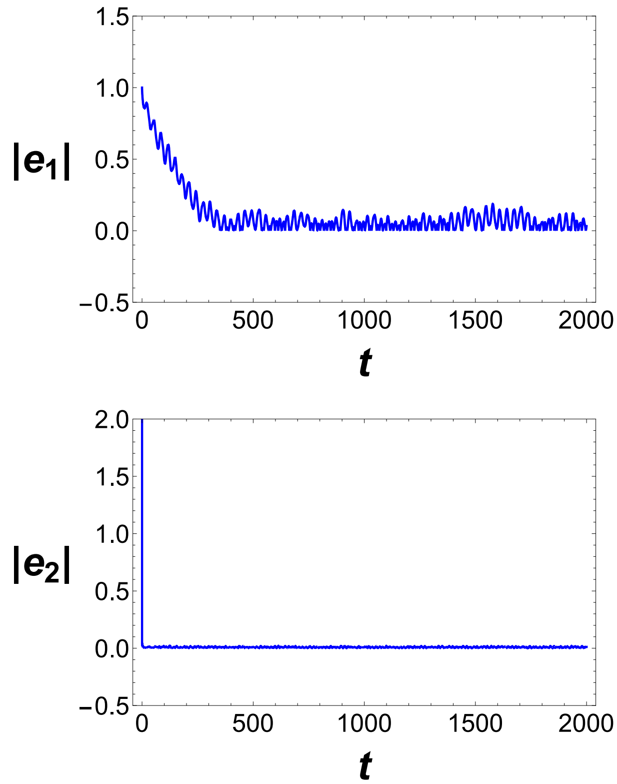

Figure 6. Error functions $|e_1|$ and $|e_2|$, for $H = 200$.

In Figure 7, it can be appreciated from a time-series plot of $x(t)$ and $y(t)$ that the signals are not in complete synchronization. On the contrary, in Figure 8 the time-series plot of $u(t)$ and $v(t)$, show that the signals are in complete synchronization.

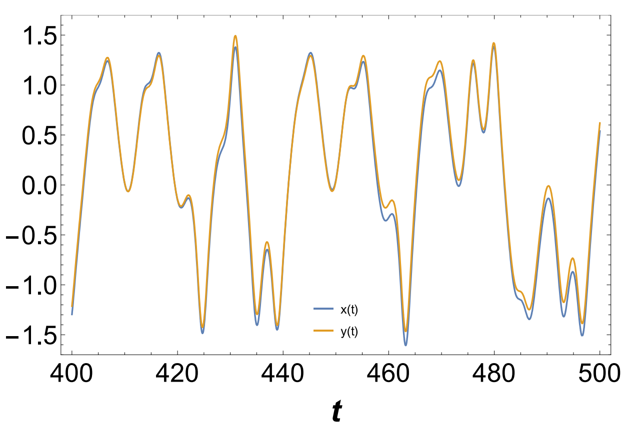

Figure 7. Time-series plot $x(t)$ and $y(t)$.

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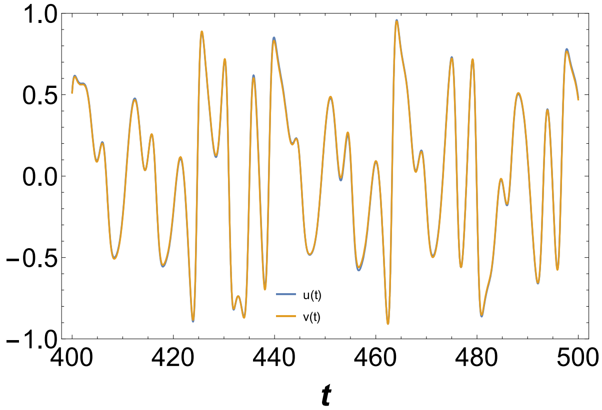

Figure 8. Time-series plot $u(t)$ and $v(t)$.

Let us analyze the projections onto the (x, y) and (u, v) planes for a specific value of $H = 200$. In this case the master system is in a chaotic regime. In Fig. 9 (a) it is shown the behavior of the Duffing oscillator (master) and in Fig. 9 (b) the Rayleigh oscillator (slave). In Fig. 9 (c) we can appreciate the fact that in the projection onto the (x, y) plane there is no complete synchronization while in the projection onto the (u, v) plane there is only complete synchronization (Fig. 9 (d)).

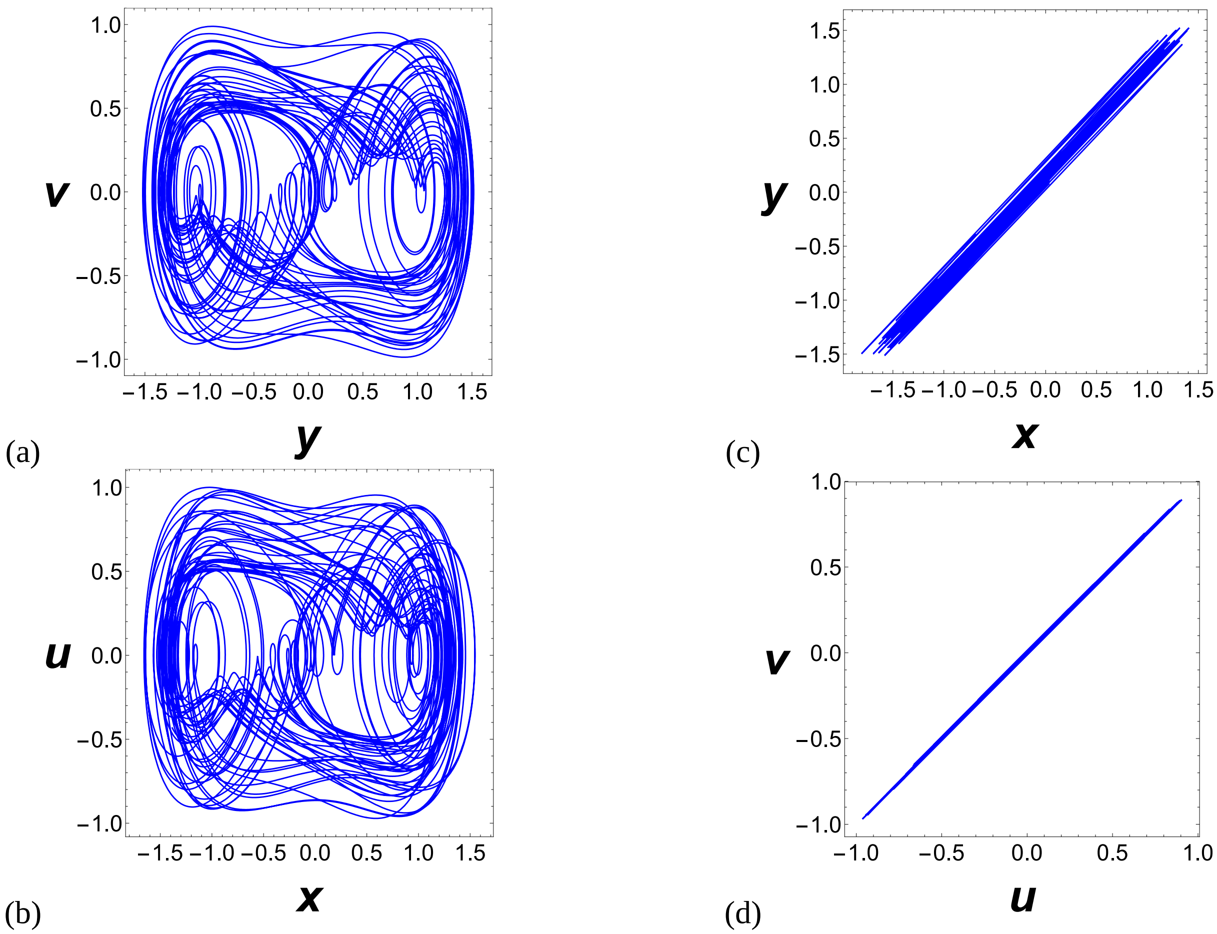

Figure 9. Dissipative coupling case, for $H = 200$. In (a) The Duffing oscillator (master) and in (b) the Rayleigh oscillator (slave). In (c) and (d) projections onto the (x, y) and (u, v) planes, respectively.

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A modified master-slave scheme leading to synchronization even in the cases where the classical master-slave scheme fails, was considered in Ref. [34]. The system analyzed in this reference can be separated in two parts (see Eqs. (13) and (14)). On one side, the system combines a non-conventional coupling, where linear feedback is made. The elastic coupling is proportional to the difference of the position, $G_1(y-x)$, which is introduced in the velocity of the slave system. The other part uses also another linear feedback proportional to the difference of the velocity (dissipative coupling), $G_2(v-u)$, introduced in the acceleration in the slave system. For the Rayleigh and Duffing oscillators, the equations that govern the evolution are

$$Master\colon \begin{cases} \dot{y} = v, \\ \dot{v} = -\alpha v + y - \varepsilon y^3 + A_2\cos(\omega_2 t), \end{cases} \quad (13)$$

$$Slave\colon \begin{cases} \dot{x} = u + G_1(y-x), \\ \dot{u} = \mu(1-u^2)u - x + A_1\cos(\omega_1 t) + G_2(v-u). \end{cases} \quad (14)$$

The errors $e_3 = y - x$ and $e_4 = v - u$, are determined by subtracting Eqs. (13) and (14), given

$$\dot{e}_3 = \dot{y} - \dot{x} = v - u - G_1 e_3,$$

$$e_4 = v - u = \dot{e}_3 + G_1 e_3,$$

$$\dot{e}_4 = \dot{v} - \dot{u} = -\alpha v + y - \varepsilon y^3 + A_2\cos(\omega_2 t)$$

$$-\mu(1-u^2)u + x - A_1\cos(\omega_1 t) - G_2(e_4). \quad (15)$$

The constant G_1 corresponds to the elastic coupling and G_2 to the dissipative coupling. Hence, $G_1(y-x) = G_1(e_1)$ and $G_2(v-u) = G_2(\dot{e}_3 + G_1 e_3)$ which manifest the dependence of G_2 on the derivative of error and the coupling G_1, giving more information about the dynamical evolution of the system. Let us introduce the vector

$$\begin{pmatrix} G_1(y-x) \\ G_2(v-u) \end{pmatrix} = \begin{pmatrix} G_1 e_3 \\ G_2\dot{e}_3 + G_2 G_1 e_3 \end{pmatrix}. \quad (16)$$

which is the called control vector, and it contains the coupling we propose. Notice that the control depends on the error and its derivative. As before, for the case $G_1 = G_2 = 0$, the system decouples. In order to study the dynamics of the system, we vary the couplings G_1 and G_2 keeping one of them constant, while the other is varied. Let us consider the $|y(t) - x(t)|$ and $|v(t) - u(t)|$ error functions. We calculate $|y(t) - x(t)|$ keeping $G_2 = 100$ and varying G_1 in small steps from 0 to 10. In a similar way, we obtain the error function $|v(t) - u(t)|$ with $G_1 = 5$ and varying G_2 in small steps from 0 to 200. As it can be appreciated from Figs. 10 and 11, we obtain complete synchronization, since the error functions go to zero as the value of G_1 and G_2 are increased. The plots of $|e_3|$ and $|e_4|$ as a function of t, for the values of $G_1 = 5$ and $G_2 = 100$, are depicted in Fig. 12.

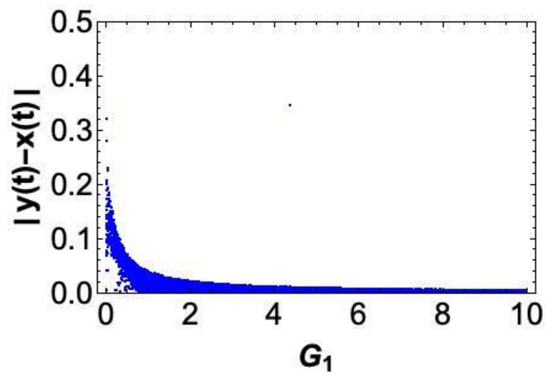

Figure 10. The error function $|y(t) - x(t)|$ varying the parameter G_1.

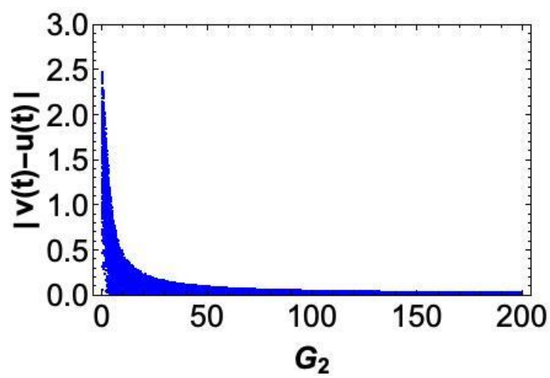

Figure 11. The error function $|v(t) - u(t)|$ varying the parameter G_2.

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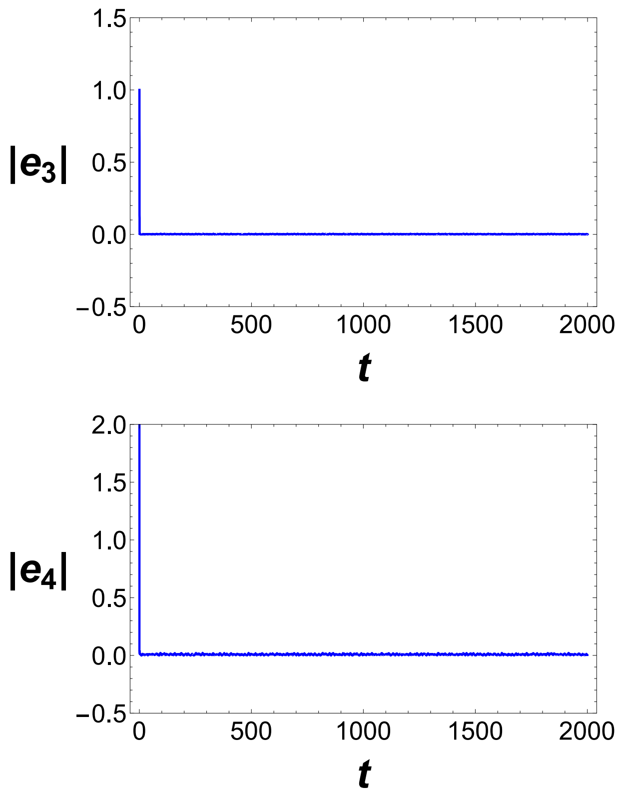

Figure 12. Error functions $|e_3|$ and $|e_4|$ with respective values of $G_1 = 5$ and $G_2 = 100$.

As it can be observed in Figs. 13 and 14, the time-series plots of $x(t)$, $y(t)$, $u(t)$, and $v(t)$ shown that the signals are in complete synchronization.

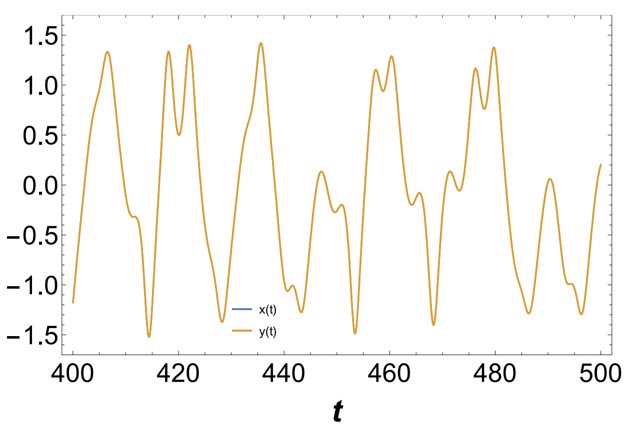

Figure 13. Time-series plot $x(t)$ and $y(t)$.

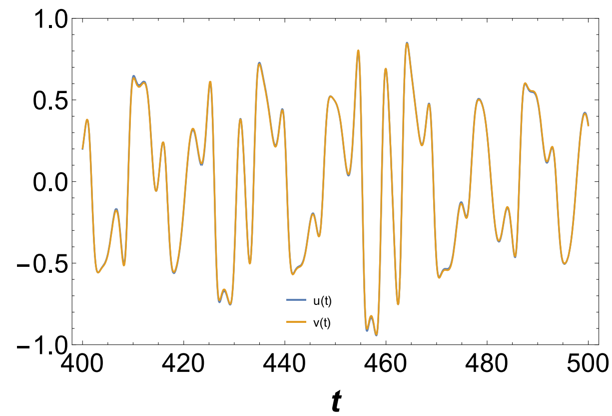

Figure 14. Time-series plot $u(t)$ and $v(t)$.

Let us now analyze the projections onto the (x, y) and (u, v) planes for values of $G_1 = 5$ and $G_2 = 100$. For this case, the Duffing oscillator is in a chaotic regime; the Rayleigh oscillator is maintained as the slave system. In Figs. 15 (a) and (b) the behavior of the Duffing and Rayleigh oscillators is shown, respectively, while in (c) and (d), it can be observed that complete synchronization is reached for these systems.

For certain systems, it is not possible to reach synchronization when the classical master-slave scheme is used. In some cases, the systems reach complete synchronization in a single state of the slave system as it occurs for the dynamics contained in Eqs. (7) and (8), depending on H, obtaining partial synchronization by using dissipative coupling for Rayleigh and duffing oscillators. Variations to the master-slave scheme for some systems have been proposed to solve certain kind of problems [44-47]. In particular, in Ref. [34] a modified master-slave scheme is considered that leads to synchronization even in the cases where the classical master-slave scheme fails.

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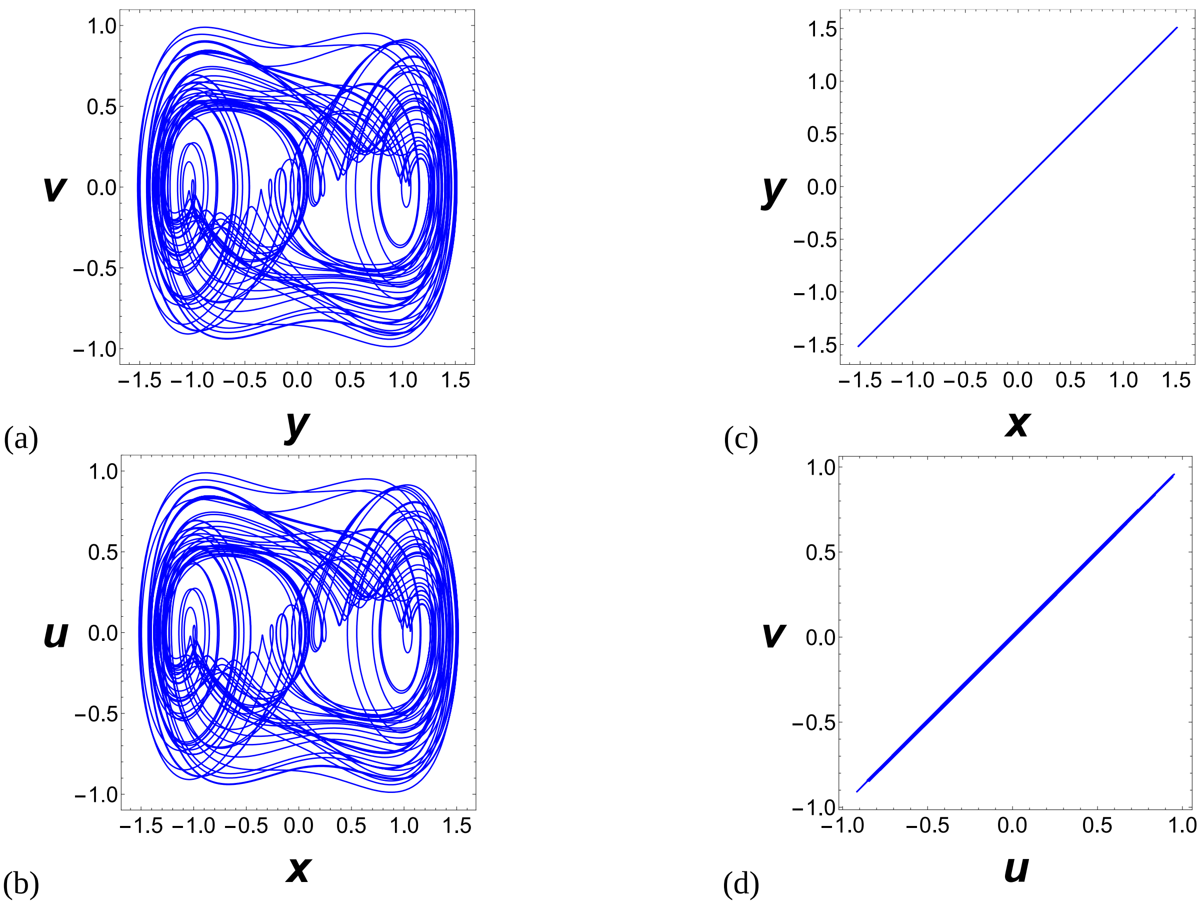

Figure 15. Elastic and dissipative couplings for $G_1 = 5$ and $G_2 = 100$. In (a) The Duffing oscillator (master) and in (b) the Rayleigh oscillator (slave). In (c) and (d) projections onto the (x, y) and (u, v) planes respectively.

4. Conclusions

The Rayleigh and Duffing are low-dimensional nonautonomous systems that present chaos and have been well studied. In this work, we have studied the master-slave configuration in the Rayleigh and Duffing oscillators, when dissipative coupling is used, only complete synchronization in the projection onto the (u, v) plane can be reached. As a matter of fact, according to the classical master-slave coupling, in the best of cases, it is obtained only complete synchronization in a single state of the slave system studied. On the other hand, the possibility of using two coupling (elastic and dissipative, in this case), blending up as one, allows the system a more interesting dynamics and a broad range for the control parameters. In this paper, we have analyzed the synchronization in the Rayleigh and Duffing oscillators using the combination of the elastic and dissipative couplings. We observed that, in a difference with other approaches, with this new coupling, we were able of obtaining complete synchronization in the projections onto the (x, y) and (u, v) planes. In order to apply synchronization in communication systems, it is necessary to have a large range of the control parameter, which is obtained in the Rayleigh and Duffing oscillators, by employing our approach of coupling. This kind of coupling will be applied in others systems that do not present synchronization through the usual methods.

ISSN: 2594-1925

5. Acknowledgements

This work has been partially supported by UMNSH. U. Uriostegui-Legorreta thanks to CONACYT for financial support.

6. Authorship acknowledgements

Ulises Uriostegui Legorreta: Conceptualización; Recursos; Ideas; Metodología; Análisis formal; Investigación; Análisis de datos; Borrador original; Administración de proyecto. *Eduardo Salvador Tututi Hernández:* Conceptualización; Ideas; Metodología; Análisis formal; Análisis de datos; Escritura; Revisión y edición.

References

[1] L.M. Pecora and T.L. Carroll, *Synchronization in chaotic systems*, Phys. Rev. Lett. 64, (1990) 821. https://doi.org/10.1103/PhysRevLett.64.821

[2] I. Pastor-Diaz and A. López-Fraguas, *Dynamics of two coupled van der Pol oscillators*, Phys. Rev. E 52, (1995) 1480. https://doi.org/10.1103/PhysRevE.52.1480

[3] C. Reick and E. Mosekilde, *Emergence of quasiperiodicity in symmetrically coupled, identical period-doubling systems*, Phys. Rev. E52, (1995) 1418. https://doi.org/10.1103/PhysRevE.52.1418

[4] M. Z. Ding and W. H. Yang, and H. J. Zhang, *Observation of intermingled basins in coupled oscillators exhibiting synchronized chaos*, Phys. Rev. E 54, (1995) 2489. https://doi.org/10.1103/PhysRevE.54.2489

[5] H-W. Yin and J-H. Dai, *Phase effect of two coupled periodically driven Duffing oscillators*, Phys. Rev. E 58, (1998) 5683. https://doi.org/10.1103/PhysRevE.58.5683

[6] K-J. Lee, Y. Kwak and T-K. Lim, *Phase jumps near a phase synchronization transition in systems of two coupled chaotic oscillators*, Phys. Rev. Lett. 81, (1998) 321. https://doi.org/10.1103/PhysRevLett.81.321

[7] L.O. Chua, M. Itoh, L. Kocarev and K. Eckert, *Chaos synchronization in Chua circuit*, J. Circuits, Systems and Computers 3, (1993) 93. https://doi.org/10.1142/S0218126693000071

[8] E.D. Dongmo, K.S. Ojo, P. Woafo and A.N. Njah, *Difference Synchronization of Identical and Nonidentical Chaotic and Hyperchaotic Systems of Different Orders Using Active Backstepping Design*, ASME. J. Comput. Nonlinear Dynam. 13, (2018) 051005. https://doi.org/10.1115/1.4039626

[9] D. López-Mancilla, G. López-Cahuich, C. Posadas-Castillo, C.E. Castañeda, J.H. García-López, J.L. Vázquez-Gutiérrez and E. Tlelo-Cuautle, *Synchronization of complex networks of identical and nonidentical chaotic systems via model-matching control*, PLoS ONE, 14, (2019) 0216349. https://doi.org/10.1371/journal.pone.0216349

[10] C. Huang and J. Cao, *Active control strategy for synchronization and anti-synchronization of a fractional chaotic financial system*, Physica A: Statistical Mechanics and its Applications, 473, (2017) 262. https://doi.org/10.1016/j.physa.2017.01.009

[11] Q. Ye, Z. Jiang and T. Chen, *Adaptive Feedback Control for Synchronization of Chaotic Neural Systems with Parameter Mismatches*, Complexity, Vol. 2018, (2018) 5431987. https://doi.org/10.1155/2018/5431987

[12] J.A. Chekan, M.A. Nojoumian, K. Merat and H. Salarieh, *Chaos control in lateral oscillations of spinning disk via linear optimal control of discrete systems*, J. Vib. Control, 23, (2017) 103. https://doi.org/10.1177/1077546315575248

[13] M.C. Pai, *Sliding Mode Control for Discrete-Time Chaotic Systems with Input Nonlinearity*, ASME: J. Dyn. Sys. Meas. Control, 142, (2020) 101003. https://doi.org/10.1115/1.4047218

[14] J. Yan and C. Li, *Generalized projective synchronization of a unified chaotic system*,

ISSN: 2594-1925

Chaos, Solitons and Fractals, 26, (2005) 1119. https://doi.org/10.1016/j.chaos.2005.02.034
[15] A.Razminia and B.Dumitru, *Complete synchronization of commensurate fractional order chaotic systems using sliding mode control*, Mechatronics, 23, (2013) 873. https://doi.org/10.1016/j.mechatronics.2013.02.004
[16] A. Ouannas, A.T. Azar and S. Vaidyanathan, *New hybrid synchronization schemes based on coexistence of various types of synchronization between master-slave hyperchaotic systems*, Int. J. Comput. Appl. Technol, 55, (2017) 112. https://doi.org/10.1504/IJCAT.2017.10003542
[17] E. Campos, J. Urias and N.F. Rulkov, *Multimodal synchronization of chaos*, Chaos: An Interdisciplinary Journal of Nonlinear Science, 14, (2004) 48-54. https://doi.org/10.1063/1.1624651
[18] J.S. González Salas, E. Campos Cantón, F.C. Ordaz Salazar and I. Campos Cantón, *Forced synchronization of a self-sustained chaotic oscillator*, Chaos: An Interdisciplinary Journal of Nonlinear Science, 18, (2008) 023136. https://doi.org/10.1063/1.2947661
[19] A. Anzo-Hernández, E. Campos-Cantón and M. Nicol, *Itinerary synchronization between PWL systems coupled with unidirectional links*, Commun Nonlinear Sci NumerSimulat, 70, (2019) 102. https://doi.org/10.1016/j.cnsns.2018.10.020
[20] A. Khan and Shikha, *Hybrid function projective synchronization of chaotic systems via adaptive control*, Int. J. Dynam. Control, 5, (2017) 1114. https://doi.org/10.1007/s40435-016-0258-6
[21] A.Karimov, A.Tutueva, T.Karimov, O.Druzhina and D.Butusov, *Adaptive Generalized Synchronization between Circuit and Computer Implementations of the Rössler System*,Appl. Sci. 2021, 11, 81. https://doi.org/10.3390/app11010081
[22] K. Murali and M. Lakshmanan, *Transmission of signals by synchronization in a chaotic Van der Pol–Duffing oscillator*, Phys. Rev. E 48, (1993) R1624. https://doi.org/10.1103/PhysRevE.48.R1624
[23] A.N. Njah, *Synchronization via active control of parametrically and externally excited* Φ^6 *Van der Pol and Duffing oscillators and application to secure communications*, Journal of Vibration and Control 17, (2010) 504. https://doi.org/10.1177/1077546309357024
[24] L. Lu, F. Zhang and C. Han, *Synchronization transmission of the target signal in the circuit network based on coupling technique*, Physica A: Statistical Mechanics and its Applications 535, (2019) 122412. https://doi.org/10.1016/j.physa.2019.122412
[25] M.S. Dutra, A.C. de Pina Filho and V.F. Romano, *Modeling of a bipedal locomotor using coupled nonlinear oscillators of Van der Pol*, Biol. Cybern 88, (2003) 292. https://doi.org/10.1007/s00422-002-0380-8
[26] R.F. Jasni and A.A. Shafie, *Van Der Pol Central Pattern Generator (VDP-CPG) Model for Quadruped Robot*, IRAM 2012. Communications in Computer and Information Science 330, (2012) 175. https://doi.org/10.1007/978-3-642-35197-6_18
[27] S. Mall and S. Chakraverty, *Hermite Functional Link Neural Network for Solving the Van der Pol–Duffing Oscillator Equation*, Neural Computation 28, (2016) 1598. https://doi.org/10.1162/NECO_a_00858
[28] S.C. Sarkhosh, S.C. Shahriar and P.P. See, *Application of Chebyshev neural network to solve Van der Pol equations*, International Journal of Basic and Applied Sciences 10, (2021) 19. https://doi.org/10.14419/ijbas.v10i1.31431
[29] J.C. Chedjou, K. Kyamakya, I. Moussa, H.P. Kuchenbecker, W. Mathis, *Behavior of a self-sustained electromechanical transducer and routes to chaos*, J. Vib. Acoust. 128, (2006) 282. https://doi.org/10.1115/1.2172255

ISSN: 2594-1925

[30] J.C. Chedjou, H.B. Fotsin, P. Woafo, and S. Domngang, *Analog simulation of the dynamics of a van der Pol oscillator coupled to a Duffing oscillator*, IEEE Trans. Circuits Syst.I, Fundam. Theory Appl. 48, (2001) 748. https://doi.org/10.1109/81.9281575

[31] A.P. Kuznetsov, N.V. Stankevich, L.V. Turukina, *Coupled van der Pol-Duffing oscillators: phase dynamics and structure of synchronization tongues*, Physica D 238, (2009) 1203. https://doi.org/10.1016/j.physd.2009.04.001

[32] M.S. Siewe, S.B. Yamgoué, E.M. MoukamKakmeni, C. Tchawoua, *Chaos controlling self-sustained electromechanical seismograph system based on the Melnikov theory*, Nonlinear Dyn. 62, (2010) 379. https://doi.org/10.1007/s11071-010-9725-3

[33] U.E. Vincent and A. Kenfack, *Synchronization and bifurcation structures in coupled periodically forced non-identical Duffing oscillator*, Phys. Scr. 77, (2008) 045005. https://doi.org/10.1088/0031-8949/77/04/045005

[34] U. Uriostegui, E.S. Tututi and G. Arroyo, *A new scheme of coupling and synchronizing low-dimensional dynamical systems*, Rev. Mex. Fis. 67, (2021) 334. https://doi.org/10.31349/RevMexFis.67.334

[35] J. Kengne, J.C. Chedjou, G. Kenne, K. Kyamakya and G.H. Kom, *Analog circuit implementation and synchronization of a system consisting of a van der Pol oscillator linearly coupled to a Duffing oscillator*, Nonlinear Dyn, 70, (2012) 2163. https://doi.org/10.1007/s11071-012-0607-8

[36] J. Kengne, F. Kenmogne and V. Kamdoum Tamba, *Experiment on bifurcation and chaos in coupled anisochronous self-excited systems: Case of two coupled van der Pol-Duffing oscillators*, Journal of Nonlinear Dynamics, 2014, (2014) 815783. https://doi.org/10.1155/2014/815783

[37] U. Uriostegui and E.S. Tututi, *Synchronization in the van der Pol-Duffing system via elastic and dissipative couplings*, Rev. Mex. Fis. 68, (2022) 011402, pp.1–13. https://doi.org/10.31349/RevMexFis.68.011402

[38] H. Zhang, D. Liu and Z. Wang, *Controlling Chaos: Suppression, Synchronization and Chaotification*, Springer, London, (2009). https://doi.org/10.1007/978-1-84882-523-9

[39] S. Boccaletti, J. Kurths, G. Osipov, DL. Valladares and CS. Zhou, *The synchronization of chaotic systems*, Physics Reports 366, (2002) 101. https://doi.org/10.1016/S0370-1573(02)00137-0

[40] L. M. Pecora and T. L. Carroll, *Synchronization of chaotic systems*, Chaos, 25, (2015) 097611. https://doi.org/10.1063/1.4917383

[41] T-P Chang, *Chaotic motion in forced duffing system subject to linear and nonlinear damping*, Mathematical Problems in Engineering, Vol. 2017, (2017) 3769870. https://doi.org/10.1155/2017/3769870

[42] M.S. Siewe, C. Tchawoua, and P. Woafo, *Melnikov chaos in a periodically driven Rayleigh-Duffing oscillator*, Mechanics Research Communications, Vol. 37, (2010) 363. https://doi.org/10.1016/j.mechrescom.2010.04.001

[43] Y-Z. Wang, and F-M. Li, *Dynamical properties of Duffing-van der Pol oscillator subject to both external and parametric excitations with time delayed feedback control*, Journal of Vibration and Control, Vol. 21, (2015) 371. https://doi.org/10.1177/1077546313483160

[44] K. Ding, *Master-Slave Synchronization of Chaotic Φ^6 Duffing Oscillators by Linear State Error Feedback Control*, Complexity Vol. 2019, (2019) 3637902. https://doi.org/10.1155/2019/3637902

[45] A. Buscarino, L. Fortuna, and L. Patane, *Master-slave synchronization of hyperchaotic systems through a linear dynamic coupling*, Phys. Rev. E 100, (2019) 032215.

ISSN: 2594-1925

https://doi.org/10.1103/PhysRevE.100.032215
[46] F. Aydogmus and E. Tosyali, *Master–slave synchronization in a 4D dissipative nonlinear fermionic system*, International Journal of Control Vol. 2020, (2020) 1. https://doi.org/10.1080/00207179.2020.1808244

[47] J.P. Ramirez, E. Garcia and J. Alvarez, *Master-slave synchronization via dynamic control*, Common Nonlinear Sci NumerSimulat, Vol. 80, (2020) 104977. https://doi.org/10.1016/j.cnsns.2019.104977

ISSN: 2594-1925

ISSN: 2594-1925